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ON THE THEORY OF SUSPENSION BRIDGES

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INTRODUCTION

It is universally assumed in the text-books on suspension bridges that the shape of the curve assumed by the cable is a *parabola* on the supposition that the weight of the structure and the loads are uniformly distributed in a horizontal direction. Neglecting for a moment the moving loads, the weight of the structure is made up of the weight of the cable and the suspenders, and the weight of the roadway and the stiffening strusses, it being assumed that the stiffening is not of the braced-chain type. We could simply classify this as the weight of the cable and the weight of the roadway, the former being uniformly distributed along its length and the latter horizontally. It is well-known in statics that if the former be inappreciable and can be neglected the curve of equilibrium is a *parabola*; while if the latter be neglected in comparison with the cable weight the curve is a *catenary*. It is hardly necessary to point out that the curve will still be a *parabola* even if the former be not neglected but taken to act horizontally. I propose to examine in this paper how far this assumption is justifiable

* Paper presented at the 26th Annual Session of the Mysore Engineers Association held in March 1934.

and what is the error made by the assumption in calculations relating to suspension bridges. In view of the fact that every book, either on applied mechanics or suspension bridges, makes this assumption without any attempt at justification, it is worthwhile to investigate how far it is correct. Before proceeding to this discussion, it might be observed that greater errors are likely to result from this assumption, the greater the weight of the cable. The following *table giving a comparative statement

Name.	Span in ft.	Net weight of chain per ft. run.	Gross fixed load per ft. of span.
Clifton	702.25	1560	3171
Freiburg	870	167	760
Niagara	821.3	820	2032

for three suspension bridges shows that the cable weight is quite appreciable when compared with the gross fixed load and it may consequently be of some practical use to undertake this investigation. Further, it is also evident that large values of the ratio of sag to span introduce greater errors in the assumption and in practice this ratio varies from 0.2 to 0.05 such that it might be as high as $1/5$.

§ 1. THE GENERAL DIFFERENTIAL EQUATION

To determine the form of the cable for any loading, consider vertical loads applied on a cable suspended between two points. The cable will assume a definite polygonal form determined by the

* Table from *Rankine*: Civil Engineering. p. 582.

relations between the loads. The end components will be inclined and will have horizontal components H . Simple considerations of static equilibrium show that H will be the same for both end reactions and will also equal the horizontal component of the tension in the cable at any point. H is called the horizontal tension of the cable. It is easily shown that the cable curve is the bending moment diagram for the applied loads. If the loads are continuously distributed the funicular polygon becomes a continuous curve and if w be the load per horizontal linear unit at any point having the abscissa x , we can obtain the equation for the vertical shear at any section x of the span and on differentiating this equation we obtain,

$$H \frac{d^2 y}{dx^2} = -w$$

and on taking the y -axis vertically upwards this becomes

$$H \frac{d^2 y}{dx^2} = w \quad (1)$$

which might be taken as the differential equation* of the equilibrium curve in the general case. Let g_0 be the weight of the cable per unit length (supposed uniform) and w_0 the total uniform moving and dead loads per horizontal foot; equation (1) then becomes

$$H \frac{d^2 y}{dx^2} = g_0 \sec \alpha + w_0 \quad (2)$$

where α is the inclination of the cable to the horizontal at any point. Eqn. (2) is the general differential equation of the problem. If g_0 be assumed to be

* See for e.g., W. H. BURR: *Suspension Bridges* p. 61. Eqn. (13) or JOHNSON, BRYAN and TURNEAURE: *Modern Framed Structures*. Part II, pp. 187-88. We shall use throughout the notation of the former book.

uniform horizontally instead of along the curve, we can reduce the equation to

$$H \frac{d^2 y}{dx^2} = w \quad (w = w_0 + g_0)$$

and taking the initial conditions at the lowest point of the cable to be :

$$x = 0, \frac{dy}{dx} = 0, y = 0, \text{ this equation gives}$$

$$Hy = \frac{wx^2}{2} \text{ or } y = \frac{wx^2}{2H}$$

which is the standard equation to the parabolic cable. The tension T at any point is given by

$$T = H \sec \alpha = H \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

If the points of support are at the elevation $y = f$ and the span $= l$, H takes the value

$$H = \frac{wl^2}{8f} \text{ and } T = \frac{wl^2}{8f} \sqrt{1 + \left(\frac{4f}{l}\right)^2}$$

If L be the entire length of the cable we have

$$\begin{aligned} L &= 2 \int_0^{l/2} \left[1 + \left(\frac{dy}{dx}\right)^2 \right]^{\frac{1}{2}} dx \frac{l}{16n} (2u + \sin hu) * \\ &= 2 \int_0^{l/2} \left[1 + \frac{64f^2}{l^4} x^2 \right]^{\frac{1}{2}} dx \end{aligned}$$

Developing the radical by the binomial theorem to three terms and performing the integration indicated

$$L = l \left\{ 1 + \frac{8}{3} \frac{f^2}{l^2} - \frac{32}{5} \cdot \frac{f^4}{l^4} \right\}$$

The equations for y , H , T , L are the fundamental equations for the parabolic cable.

* $n = f/l$ and u is defined by $\sin hu = 4n$. This elegant form for L is to be found in Steinman; *Suspension Bridges* p. 5.

If in equation (2) we neglect w_0 we obtain

$$H \frac{d^2 y}{dx^2} = g_0 \sec \alpha = g_0 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

This is the differential equation of a catenary and its solution gives

$$y = \frac{1}{2c} \left(e^{cx} + e^{-cx} - 2 \right) \text{ where } c = g_0/H$$

and we can derive corresponding equations for H, T and L as in the parabolic cable.

I now consider the general differential equation (2) where neither w_0 nor g_0 can be neglected (it must be observed that assuming a uniform horizontal distribution of g_0 is mathematically equivalent to neglecting it) and obtain the exact mathematical solution. In the next article, I derive an approximate form of the solution suitable for practical use and enabling us to compare the formulæ obtained for a strictly parabolic cable.

We shall write (2) in the form

$$H \frac{d^2 y}{dx^2} = g_0 \sec \Theta + w_0 \text{ (replacing } \alpha \text{ by } \Theta \text{)}$$

$$\text{or } \frac{d^2 y}{dx^2} = a \sec \Theta + b. \quad \left(a = \frac{g}{H}, b = \frac{w}{H} \right)$$

and assume, as always realised in practice, that $b > a$. Putting $p = dy/dx$, this further reduces to the first order differential equation:

$$\frac{dp}{dx} = a \sqrt{1 + p^2 + b}. \quad (3)$$

Since $p = \tan \Theta$, $dp = \sec^2 \Theta \cdot d\Theta$ and (3) can be written as

$$\frac{\sec^2 \Theta \cdot d\Theta}{b + a \sec \Theta} = dx, \text{ or } \frac{d\Theta}{\cos \Theta (a + b \cos \Theta)} = dx$$

$$i. e., \frac{d\Theta}{\cos \Theta} - \frac{bd\Theta}{a + b \cos \Theta} = adx$$

and integrating we have

$$ax = \log (\sec \Theta + \tan \Theta) -$$

$$\sqrt{\frac{b}{b^2 - a^2}} \log \left\{ \frac{b + a \cos \Theta + \sqrt{b^2 - a^2} \sin \Theta}{a + b \cos \Theta} \right\} \quad (4)$$

the constant of integration being given by $x = 0$ for $\Theta = 0$.

Again $\frac{d p}{d x} = a \sqrt{1 + p^2} + b$ can be written as

$$\frac{p dp}{a \sqrt{1 + p^2} + b} = dy \text{ or } \frac{\tan \Theta \sec^2 \Theta d \Theta}{a \sec \Theta + b} = dy$$

This can be easily integrated and the result is

$$a^2 y = a (\sec \Theta - 1) + b \log \left[\frac{a + b}{b + a \sec \Theta} \right] \quad (5)$$

taking $y = 0$ for $\Theta = 0$.

Equations (4) and (5) give the parametric equations of the curve of the cable in terms of Θ , but they are in quite an unmanageable form for any practical purposes. Before deriving approximate equations from these, it is instructive to see how these reduce to the parabola and catenary for the cases $a=0$ and $b=0$ respectively.

Case $b=0$: Equation (4) takes the simple form

$$ax = \log (\sec \Theta + \tan \Theta)$$

and equation (5) reduces to

$ay = \sec \Theta - 1$, since the argument of the logarithm in equation (5) can never become infinite. Eliminating Θ between these equations we have

$$y = \frac{1}{2a} \left(\frac{ax}{e} - \frac{ax}{e} - 2 \right)$$

which is the equation to the catenary

Case $a = 0$: In this case equations (4) and (5) reduce to identities but by dividing them throughout by a and a^2 respectively and applying de L'Hospital's Rule for the evaluation of indeterminate forms, they take the form

$$\left. \begin{aligned} &= \frac{1}{b} \tan \Theta \\ \text{and } y &= \frac{1}{2b} \tan^2 \Theta \end{aligned} \right\}$$

which give $y = \frac{bx^2}{2}$, the equation to the parabola.

§ 2. APPROXIMATE SOLUTION OF THE GENERAL DIFFERENTIAL EQUATION.

The differential equation (3) is equivalent to the two equations

$$\frac{dy}{dx} = \frac{z}{\sqrt{1-z^2}}$$

$$\text{and } \frac{dz}{dx} = a(1-z^2) + b(1-z^2)^{3/2}$$

where $z = \sin \Theta$. Assuming for y and z two power series* with undetermined coefficients, using the initial conditions $x=0$, $y=0$, $z=0$ and determining these coefficients so as to satisfy the above two equations, we deduce for y the expression

$$y = \frac{(a+b)x^2}{2} + \frac{ab^2x^4}{24} + \quad (5)$$

* This method is suggested for approximate solutions of differential equations in RUNGE UND KÖNIG: *Numerisches Rechnen*. § 87. pp 321—23

As a first approximation we can retain only the second term in (5), and the equation then illustrates how the assumption of non-uniform horizontal distribution of g_0 affects the shape of the parabolic cable.

I shall now indicate an alternative method of deriving equation (5) which is more illuminating. Equation (2), *viz.*

$$\frac{dp}{dx} = a \sqrt{1+p^2} + b \quad (2)$$

gives us the parabolic cable when we write a in place of $a \sqrt{1+p^2}$ which is equivalent to putting $p=0$ *i.e.*, treating the cable as straight. We can therefore obtain a better approximation by putting for $p = \left(\frac{dy}{dx}\right)$ its value as obtained for a parabolic cable. From § 1, the equation

$y = \frac{b x^2}{2}$ gives $p = \frac{dy}{dx} = bx$ and hence (2) can be approximated to

$$\begin{aligned} \frac{dp}{dx} &= a (1 + b^2 x^2)^{\frac{1}{2}} + b \\ \text{i.e., } \frac{d^2 y}{dx^2} &= a \left\{ 1 + \frac{b^2 x^2}{2} - \frac{b^4 x^4}{8} - \dots \right\} + b, \end{aligned}$$

on expanding by the binomial theorem, since bx giving the slope at any point can be assumed, for all practical purposes, numerically less than unity, and the expansion being therefore valid. Retaining only two terms, this gives

$$\frac{d^2 y}{dx^2} = (a+b) + \frac{ab^2 x^2}{2}$$

Integrating once and using $\frac{dy}{dx} = 0$ if $x = 0$

$$\frac{dy}{dx} = (a+b)x + \frac{ab^2 x^3}{6}$$

Another integration, using $y = 0$ if $x = 0$ gives

$$y = \frac{(a + b) x^2}{2} + \frac{ab^3 x^4}{24} \quad (5)$$

which is the same * as equation (5). For use in practical calculations it is preferable to express this equation in terms of f and l ; but we have only one condition expressing that (5) passes through the points of support ($\pm l/2, f$) and this does not enable the determination of both a and b in terms of these quantities. Let us, however, write $a = bn$, so that n is the fraction of the cable weight to the horizontal load; equation (5) then reduces to

$$y = \frac{b (n + 1) x^2}{2} + \frac{nb^3 x^4}{24}$$

Putting $x = l/2$ & $y = f$, we have

$$f = \frac{b (n + 1) l^2}{8} + \frac{nb^3 l^4}{384}$$

$$\text{i.e., } f - \frac{bl^2}{8} = \frac{nb l^2}{8} \left(1 + \frac{b^2 l^2}{48} \right). \quad (6)$$

This equation gives a cubic in b from which we can find b in terms of n, f and l which might be supposed to be known quantities. We can solve this cubic equation approximately.

Writing $\frac{bl^2}{8} = \Delta$, eqn. (6) reduces to

$$f - \Delta = n\Delta \left(1 + \frac{4\Delta^2}{3l^2} \right)^2, \text{ a cubic in } \Delta$$

For small values of n , a first approximation is

$$f = \Delta \text{ \& \; } b = \frac{8f}{l^2}$$

* See W. H. BURR: *ibid* p. 69 where a similar method is adopted in a different connection.

which is a well-known expression in a parabolic cable. To obtain a second approximation let us substitute this value $\Delta = f$ in the right hand side and obtain

$$\Delta = f - nf \left(1 + \frac{4}{3} \frac{f^2}{l^2} \right) = f \left(1 - n - \frac{4n}{3} \frac{f^2}{l^2} \right)$$

$$i. e., b = \frac{8f}{l^2} \left(1 - n - \frac{4n}{3} \frac{f^2}{l^2} \right) \quad (7)$$

$$\text{Also, } \frac{dy}{dx} = b (n+1) x + \frac{n b^3 x^3}{6} \quad (8)$$

b being given from (7) in terms of n, f and l .

Let us now calculate the length of arc of the entire cable as modified by the above formulæ and compare the same with the original expression

$$L = l \left\{ 1 + \frac{8}{3} \frac{f^2}{l^2} - \frac{32}{5} \frac{f^4}{l^4} \right\}$$

obtained in § 1. Denoting the modified length as L' we have

$$\begin{aligned} L' &= 2 \int_0^{l/2} \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{\frac{1}{2}} dx; \frac{dy}{dx} \text{ being given from (8)} \\ &= \int_0^{l/2} \left[1 + \left\{ (a+b) x + \frac{ab^3 x^3}{6} \right\}^2 \right]^{\frac{1}{2}} dx \end{aligned}$$

Expanding by the binomial theorem and retaining terms up to x^4 , this reduces to

$$\begin{aligned} L' &= 2 \int_0^{l/2} \left[1 + \frac{(a+b)^2 x^2}{2} + \left\{ \frac{(a+b) 4ab^2 - 3(a+b)^3}{24} \right\} x^4 \right] dx \\ &= l \left[1 + \frac{(a+b)^2 l^2}{24} + \frac{(a+b)}{6} \left\{ \frac{4ab^2 - 3(a+b)^3}{1920} \right\} l^4 \right] \end{aligned}$$

on performing the integration indicated. Now, to substitute $a = bn$ and $b = \frac{8f}{l^2} \left(1 - n - \frac{4nf^2}{3l^2} \right)$ in this expression, we can write

$$b^2 = \frac{64f^2}{l^4} \left(1 - n - \frac{4nf^2}{3l^2} \right)^2 \\ = \frac{64f^2}{l^4} \left(1 - 2n - \frac{8nf^2}{3l^2} \right), \text{ retaining only first power of } n,$$

$$\text{Similarly } b^4 = \frac{64^2}{l^8} \left(1 - 4n - \frac{16nf^2}{3l^2} \right)$$

$$\text{Now } L' = l \left[1 + \frac{b^2 l^2}{24} (1 + 2n) - \frac{b^4 l^4}{1920} (1 + n) \right. \\ \left. \cdot (3n^3 + 3n^2 - n + 3) \right]$$

$$= l \left\{ 1 + \frac{b^2 l^2}{24} (1 + 2n) - \frac{b^4 l^4}{1920} (3 + 2n) \right\}$$

neglecting terms in n^2 , n^3 , etc.

$$\text{i.e., } = l \left\{ 1 + \frac{b^2 l^2}{24} - \frac{b^4 l^4}{1920} + \frac{nb^2 l^2}{12} \left(1 - \frac{b^2 l^2}{80} \right) \right\}$$

Substituting in this the values of b^2 and b^4 written above, we get finally

$$L' = l \left\{ 1 + \frac{8f^2}{3l^2} - \frac{32f^4}{5l^4} + \frac{148}{9} n \frac{f^4}{l^4} + \frac{512}{5} n \frac{f^6}{l^6} \right\}$$

$$\text{Hence } L' - L = l \frac{148}{9} n \frac{f^4}{l^4} \text{ (neglecting the last term)}$$

In practice it is found that all ratios of f/l will be between .2 and .05 and the following table * gives values of L/l corresponding to different values of f/l .

f/l	L/l
.2	1.09643
.175	1.07566
.15	1.05676
.125	1.040104
.1	1.026027
.075	1.014797
.05	1.006627

* Vide *Burr* : *ibid.* (p. 73)

$$\text{The above expression } \frac{L' - L}{l} = \frac{148}{9} n \cdot \frac{f^4}{l^4}$$

gives us an indication of the error committed in treating g_0 as a uniform horizontal load. Taking from the table in the Introduction the case of the Freiburg suspension bridge for which $n = \frac{1}{5}$ (approximately) let us calculate the deviation for the case $f/l = .2$; we easily find

$$\frac{L' - L}{l} = .005262$$

such that the corrected value would be 1.10169, showing that although the modification required is not great, it is of the same order as the difference between the values of L/l calculated as in the above table and the value calculated from the formula † exact for a parabolic cable, *viz.*

$$L/l = \frac{1}{16n} (2u + \sin h2u)$$

† Vide *Burr* : *ibid.* (p. 7)

§ 3. OTHER DIFFERENTIAL EQUATIONS

In the previous investigation we have considered the weight of the cable alone and neglected that of the suspenders. Let g' be the weight of the suspenders per square unit of vertical plane of the cable so that $g'y$ is weight per linear horizontal foot of that part of the suspenders above the vertex of the cable. The part of the suspenders below the vertex forms a uniform load over the span and might be assumed to have been included in W_0 . The general differential equation becomes

$$\frac{d^2y}{dx^2} = a \sqrt{1 + p^2} + b + cy$$

$$\text{i.e., } \frac{dp}{dx} = a \sqrt{1 + p^2} + b + cy \quad (c = g'/H)$$

Making p and y the dependent and independent variables respectively, this can be reduced to the first order differential equation.

$$p \frac{dp}{dy} = a \sqrt{1 + p^2} + b + cy$$

of which an exact solution is complicated. It can however be solved by approximate methods by substituting for y and $\frac{dy}{dx}$ the expressions as given by the parabolic cable.

Putting $y = \frac{bx^2}{2}$ and $\frac{dy}{dx} = bx$, equation (10) reduces to

$$\begin{aligned} \frac{d^2y}{dx^2} &= a \sqrt{1 + b^2 x^2} + b + \frac{cbx^2}{2} \\ &= a \left(1 + \frac{b^2 x^2}{2} \right) + b + \frac{cbx^2}{2} \\ &= (a + b) + \frac{bx^2}{2} (b + c) \end{aligned}$$

Successive integrations give

$$\frac{dy}{dx} = (a+b)x + \frac{bx^3(a+b+c)}{6}$$

$$y = \frac{(a+b)x^2}{2} + b \frac{(a+b+c)x^4}{24}$$

which expresses the deviation from the parabolic form.

I now consider, finally the case where the cable is not necessarily uniform. Let g_0 be the weight per ft. run at the vertex and g_x at any point where inclination of the cable to the horizontal is Θ , supposing that the normal section of the cable at any point is proportional to its total stress, we have

$$g_x = g_0 \sec \Theta \times \sec \Theta = g_0 \sec^2 \Theta.$$

and the differential equation becomes

$$\frac{d^2y}{dx^2} = a \sec^2 \Theta + b + cy \quad (11)$$

$$\text{Putting } \frac{dy}{dx} = \frac{8fx}{l^2} \text{ and } y = \frac{4f}{l^2} x^2$$

this can be written

$$\frac{d^2y}{dx^2} = (a+b) + \left(\frac{64f^2}{l^4} + \frac{4cf}{l^2} \right) x^2$$

$$\text{giving } y = \frac{x^2}{2} (a+b) + kx^4$$

$$\text{where } k = \frac{1}{12} \left(\frac{64f^2}{l^4} + \frac{4cf}{l^2} \right)$$

An exact solution of equation (11) is a matter of some difficulty.

I hope to return in a further paper to more detailed discussions of equations (10) and (11) and also discuss, if any, the bearing of the results of § 2 on the problems of stiffening of the structure and the deflection of the bridge system.

